J. Fluid Mech. (1982), *vol.* 115, *pp.* 525–532
Printed in Great Britain

Two-dimensional periodic permanent waves in shallow water

By P. J. BRYANT
Mathematics Department, University of Canterbury, Christchurch, New Zealand

(Received 29 January 1981 and in revised form 26 May 1981)

When two periodic permanent wave trains on shallow water intersect obliquely, the regions of intersection are two-dimensional waves of permanent shape. This shape varies from a nearly linear superposition of the two wave trains at large angles of intersection between the wave normals, to a structure predominantly transverse to the direction of propagation at small angles of intersection. The latter shape is found to be governed by the two-dimensional Korteweg–de Vries (Kadomtsev–Petviashvili) equation. The two-dimensional permanent waves are stable to periodic disturbances parallel to their direction of propagation, but are unstable to certain oblique periodic disturbances.

1. Introduction

Modern techniques for solving nonlinear differential equations have been applied with success to the problem of interacting solitary waves (reviewed by Miles 1980). The interaction is weak at large angles of intersection between the wave normals (Benney & Luke 1964), when it is described in magnitude by the product of the amplitude parameters. The interaction becomes stronger at smaller angles of intersection (Miles 1977*a*), until at a certain small angle resonance occurs between the two incident waves and the resulting wave (Miles 1977*b*). The analytical solution for the interaction is singular at angles less than the resonance angle. The present investigation is concerned with the shape and properties of the two-dimensional waves formed by two periodic wave trains intersecting at angles at and below the resonance angle.

In the equivalent problem of reflection of a solitary wave by a vertical wall, experiments (Melville 1980) at small angles between the wave normal and the wall reveal that the apex of the incident and reflected waves moves away from the wall, being joined to the wall by a third wave named the Mach stem. Melville found that as the angle between the wave normal and the wall is decreased, regular reflection is replaced by Mach reflection at about the resonance angle predicted by Miles. Melville identified a problem with the momentum balance at the end of the growing reflected wave, but this difficulty is irrelevant to the present investigation.

The method of analysis used below is to describe the water surface by a double Fourier series constructed from the linear harmonics, with amplitudes varying slowly with time as a result of nonlinear interactions. A set of first-order nonlinear differential equations for the rates of change of the Fourier amplitudes is obtained. This reduces to a set of nonlinear algebraic equations for the Fourier amplitudes when the amplitudes are steady in a moving frame of reference. The set is solved numerically by the Newton–Raphson method.

A model equation has been proposed by Kadomtsev & Petviashvili (1970) which is applicable to waves on shallow water having a weak spatial variation transverse to the direction of propagation. Johnson (1980) named this equation the 'nearly plane' Korteweg–de Vries equation, and showed how it is related to other forms of the Korteweg–de Vries equation for waves on shallow water. Two-dimensional permanent-wave solutions of this equation are obtained here by the same method of analysis as is described above. These are found to agree well with the direct permanent-wave solutions when the spatial variation is weak. Krichever & Novikov (1978) have found solutions of the Kadomtsev–Petviashvili equation by methods of algebraic geometry. It has not been possible to relate their solutions to those found here.

A linear-stability analysis is made of the solutions obtained for two-dimensional permanent waves at small angles of intersection of the original periodic wave trains. The stability properties are found to be similar in form to those of one-dimensional permanent waves, with instability occurring only for certain periodic disturbances oblique to the propagation direction of the permanent waves.

2. Permanent waves

Periodic waves of wavelength $2\pi l$ along and $2\pi L$ transverse to the direction of wave propagation are generated in water of mean depth h bounded above by a free surface and below by a smooth horizontal bed. The principal non-dimensional ratios are $\epsilon = a/h$, $\mu = h/l$, and $\delta = l/L$, where a is a measure of wave amplitude. The horizontal co-ordinates x_1 and x_2 in the mean free surface are measured in units of l and L respectively, the vertical co-ordinate y in units of h, and time t in units of l/c_0 where $c_0 = (gh)^{\frac{1}{2}}$ is the linear long-wave velocity. The governing equations for the non-dimensional surface displacement $\eta(x_1, x_2, t)$ and velocity potential $\phi(x_1, x_2, y, t)$, with the upper boundary conditions expressed as perturbation expansions in ϵ, are

$$\phi_{x_1 x_2} + \delta^2 \phi_{x_2 x_2} + (1/\mu^2)\,\phi_{yy} = 0 \quad (-1 < y < 0), \tag{2.1a}$$

$$\phi_y = 0 \quad (y = -1), \tag{2.1b}$$

$$\eta_t - (1/\mu^2)\,\phi_y + \epsilon(\eta\phi_{x_1})_{x_1} + \epsilon\delta^2(\eta\phi_{x_2})_{x_2} = O(\epsilon^2) \quad (y = 0), \tag{2.1c}$$

$$\eta + \phi_t + \tfrac{1}{2}\epsilon(\phi_{x_1}^2 + \delta^2\phi_{x_2}^2 + \{1/\mu^2\}\,\phi_y^2) + \epsilon\eta\phi_{yt} = O(\epsilon^2) \quad (y = 0). \tag{2.1d}$$

Permanent-wave solutions are sought, of the form

$$\eta = \sum_{k_1=0}^{\infty} \sum_{k_2=-\infty}^{\infty} a(\mathbf{k}) \cos(k_1 x_1 + k_2 x_2 - k_1 ct), \tag{2.2}$$

with a corresponding expression for ϕ, when it may be shown that

$$\{k_1 c - \omega(\mathbf{k})\}\, a(\mathbf{k}) = \tfrac{1}{2}\epsilon \sum_{\mathbf{l}} R(\mathbf{k}, -\mathbf{l})\, a(\mathbf{l})\, a(\mathbf{k}-\mathbf{l}) + \epsilon \sum_{\mathbf{l}} R(\mathbf{k}, \mathbf{l})\, a(\mathbf{l})\, a(\mathbf{k}+\mathbf{l}) + O(\epsilon^2), \tag{2.3}$$

where $\mathbf{k} = (k_1, \delta k_2)$, $k = (k_1^2 + \delta^2 k_2^2)^{\frac{1}{2}}$, $\omega(\mathbf{k}) = \{(k/\mu)\tanh k\mu\}^{\frac{1}{2}}$ and the derivation and interaction coefficients are described in the appendix. The wave velocity c and the harmonic amplitudes $a(\mathbf{k})$ are the solutions of the set of nonlinear equations (2.3) together with the kinematic equation arising from the definition of ϵ. The latter equation is taken arbitrarily to be

$$\eta(ct, 0, t) - \eta(ct + \tfrac{1}{2}\pi, 0, t) = 2. \tag{2.4}$$

The wavelengths in the x_1 and x_2 directions may be written $-\pi \leqslant x_1 - ct \leqslant \pi$, $-\pi \leqslant x_2 \leqslant \pi$ respectively. The geometry is then such that along each of the principal diagonals $x_1 - ct = \pm x_2$, there are two wavelengths as $x_1 - ct$, x_2 traverse the above intervals. This means that

$$a(\mathbf{k}) = a(k_1, \delta k_2) = 0 \quad (k_1 \pm k_2 \,\text{odd}). \tag{2.5}$$

Symmetric two-dimensional permanent waves, such as occur when two equal wave trains intersect, also satisfy

$$a(k_1, -\delta k_2) = a(k_1, \delta k_2). \tag{2.6}$$

Symmetric-wave solutions of the above set of algebraic equations (2.3) are obtained numerically by a generalized Newton–Raphson method. Solutions are found first for a given number of harmonics $0 \leqslant k_1 \leqslant n_1$, $-n_2 \leqslant k_2 \leqslant n_2$, then n_1 and n_2 are increased step by step until the solution is unchanged to 4 decimal places by the addition of further harmonics. Non-symmetric-wave solutions, such as result from the intersection of two unequal wave trains, were not calculated because of problems with computer capacity when (2.6) was not applicable.

3. Kadomtsev–Petviashvili equation

Kadomtsev & Petviashvili (1970) proposed a model equation for solitary waves in weakly dispersing media. This KP equation, with the present notation, is

$$(\eta_t + \eta_{x_1} + \tfrac{3}{2}\epsilon\eta\eta_{x_1} + \tfrac{1}{6}\mu^2\eta_{x_1x_1x_1})_{x_1} + \tfrac{1}{2}\delta^2\eta_{x_2x_2} = 0. \tag{3.1}$$

It is the sum of the zero- and first-order terms in a perturbation expansion in the three small independent parameters $\epsilon, \mu^2, \delta^2$ for waves progressing in the forward x_1 direction (Johnson 1980). The dispersion relation satisfied by the linear terms of this equation contains the zero- and first-order terms in the perturbation expansion in μ^2, δ^2 (3.2) of the full linear dispersion relation (following (2.3)).

When wave solutions of (3.1) with the same form as (2.2) are sought, the nonlinear equations governing $a(\mathbf{k})$ are the same as (2.3) except that the frequency $\omega(\mathbf{k})$ is replaced by

$$k_1 - \tfrac{1}{6}\mu^2k_1^3 + \tfrac{1}{2}\delta^2k_2^2/k_1, \tag{3.2}$$

and the interaction coefficients R are replaced by

$$R(\mathbf{k}, \mathbf{l}) = \tfrac{3}{4}k_1. \tag{3.3}$$

Symmetric-wave solutions of the set of nonlinear equations are obtained by the same procedure as that described in § 2.

4. Examples

Three examples are illustrated in figures 1, 2 and 3, each with $\epsilon = 0{\cdot}05$ and $\mu = 0{\cdot}1$ but with three different values of δ. Each is a solution of the nonlinear equations in § 2 where no restriction is placed on the values of μ and δ. The corresponding solutions of the Kadomtsev–Petviashvili equation have been calculated, but only the first example at the largest value of δ exhibited any significant differences from the figures. The central peak of the KP solution in this example is higher and sharper than in the

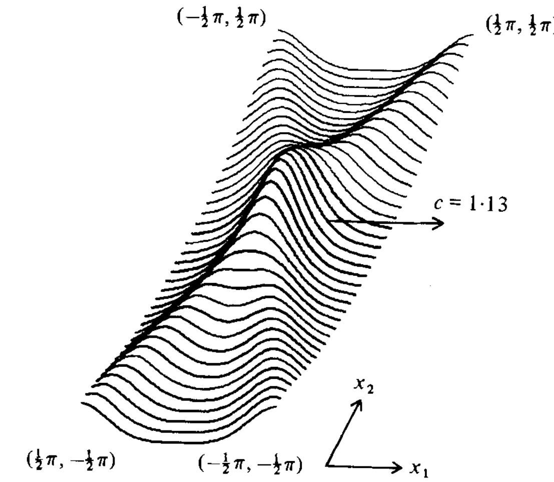

FIGURE 1. Permanent wave at $\epsilon = 0{\cdot}05$, $\mu = 0{\cdot}1$, $\delta = \tan 0{\cdot}15\pi = 0{\cdot}51$. Vertical magnification 100π.

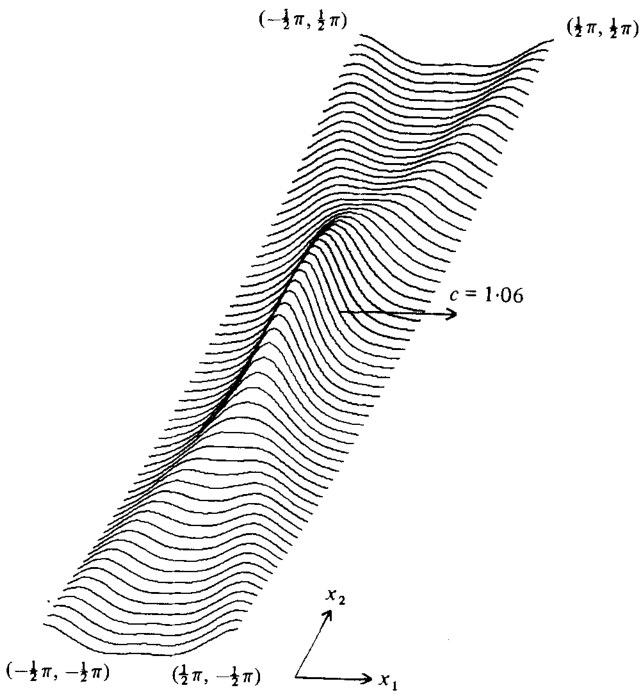

FIGURE 2. Permanent wave at $\epsilon = 0{\cdot}05$, $\mu = 0{\cdot}1$, $\delta = \tan 0{\cdot}1\pi = 0{\cdot}32$. Vertical magnification 100π.

corresponding direct solution, and the KP wave travels faster. These features of the KP solutions become more pronounced at larger values of δ than those used in the figures.

The difference between the KP solutions and the direct solutions in figures 2 and 3 is less than 1 % in each example. It should be noted that the example in figure 1 is at about the lowest value of δ, for the given ϵ and μ, for which a central stem transverse

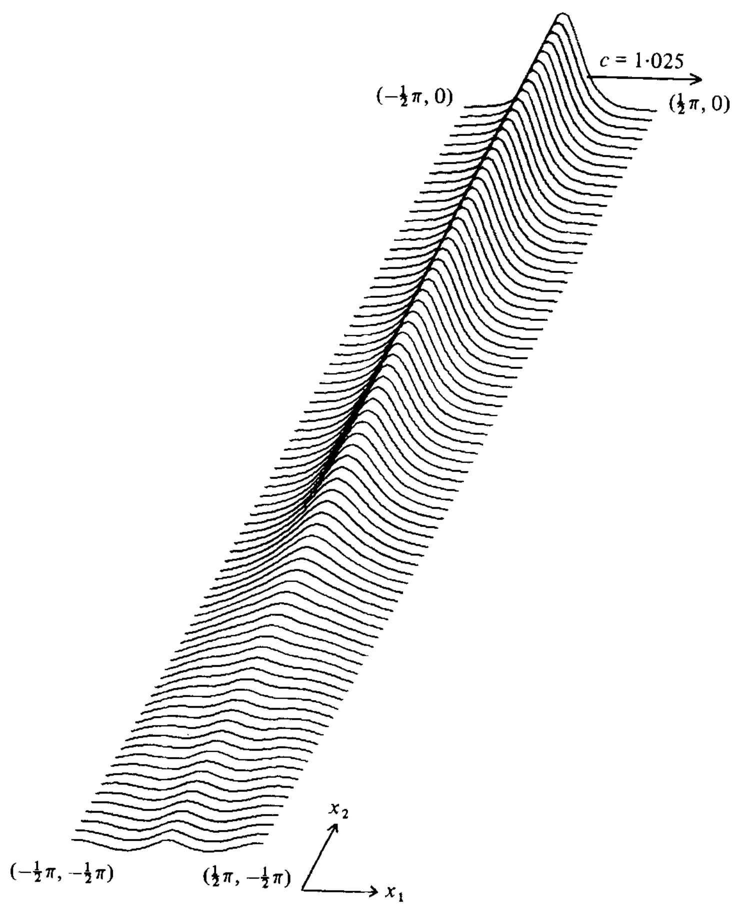

FIGURE 3. Half of the permanent wave at $\epsilon = 0{\cdot}05$, $\mu = 0{\cdot}1$, $\delta = \tan 0{\cdot}04\pi = 0{\cdot}13$. Vertical magnification 100π.

to the direction of propagation is not apparent in the figure. At the lower values of δ illustrated in figures 2 and 3 a central stem occurs, when the KP solution is in good agreement with the direct solution. Miles (1977*a*) gave a resonance criterion for the dividing line between regular and singular interactions of two oblique solitary waves. It is not possible to apply this criterion exactly to the examples illustrated because the amplitudes of the interacting waves cannot be defined unambiguously. If the amplitude is taken to be that at the four corners of each of the figures, the division between regular and singular interactions lies between the examples in figures 1 and 2. Miles' criterion provides an approximate dividing line between permanent waves with and without a central stem, corresponding respectively to permanent waves which are and which are not modelled satisfactorily by the KP equation.

As δ decreases towards zero, the shape of the two-dimensional permanent waves tends towards that of one-dimensional permanent waves transverse to the direction of propagation. The wave velocity decreases as δ decreases, tending towards that of one-dimensional waves. The two-dimensional permanent-wave solutions of the KP equation do appear to tend uniformly to the one-dimensional permanent-wave solutions of the KdV equation as δ tends to zero.

As μ decreases towards zero, the effective width of the wave in the x_1 direction is a decreasing proportion of the wavelength $2\pi l$ in this direction, but the effective length of the wave in the x_2 direction remains an almost constant proportion of the wavelength $2\pi L$ in this direction. The two-dimensional periodic permanent waves do not appear to tend towards any form of two-dimensional solitary wave lump as the wavelength tends to infinity in all directions.

5. Wave stability

The linear stability of the wave solutions may be investigated by making the time-dependent Fourier amplitudes $A(\mathbf{k})$ equal to the sum of the permanent-wave amplitudes $a(\mathbf{k})$ and a small perturbation to them, substituting in the evolution equations (A 6) for $A(\mathbf{k})$, linearizing in the small perturbations, and seeking normal-mode solutions of the set of first-order linear differential equations governing the small perturbations. The results of the calculations are similar in form to those obtained previously (Bryant 1978) for one-dimensional periodic waves in shallow water. The two-dimensional waves are stable to periodic perturbations in the direction of propagation (x_1 axis), but are unstable to perturbations in a narrow band of wavenumbers oblique to the direction of propagation.

The explanation for the region of instability appears to be the same as that advanced in § 5 of Bryant (1978), namely that the sum of the side-band disturbance harmonics $\mathbf{k}-\boldsymbol{\kappa}, \mathbf{k}+\boldsymbol{\kappa}$, where $\mathbf{k} = (k_1, 0)$, interacts resonantly with the harmonic of the permanent wave with wavenumber $2\mathbf{k}$. Resonance occurs in the neighbourhood of disturbance harmonics $\boldsymbol{\kappa}$ satisfying

$$\omega(\mathbf{k}-\boldsymbol{\kappa})+\omega(\mathbf{k}+\boldsymbol{\kappa}) = 2\mathbf{k}.\mathbf{c} = 2k_1 c. \tag{5.1}$$

The solution of (5.1) is found to be close to the region of instability in all examples calculated, agreement being improved by allowing for the effect of forward particle velocities in the wave itself. The unstable growth is caused by the application to the permanent waves of a periodic modulation with a length scale large compared with $2\pi l$ ($\kappa \ll k_1$), at an angle to the direction of propagation near that determined by (5.1). The modulation grows initially with a time scale inversely proportional to ϵ^2. McLean (1982) describes a range of instabilities of one-dimensional permanent waves in shallow water, of which the above type of instability is the first.

Appendix

The derivation of the set of equations (2.3) from the set (2.1) is summarized. When solutions to the set (2.1) are sought with the form

$$\eta = \tfrac{1}{2}\sum_{\mathbf{k}} A(\mathbf{k}) \exp i\{\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t\}+*, \tag{A 1}$$

$$\phi = \tfrac{1}{2}\sum_{\mathbf{k}} B(\mathbf{k}) \cosh \mu k(1+y) \exp i\{\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t\}+*, \tag{A 2}$$

(with the notation of § 2, and $x = (x_1, x_2)$, $*$ denotes complex conjugate) the linear solution, in which all amplitudes A and B are constants, is

$$A(\mathbf{k})-i\omega(\mathbf{k}) \cosh \mu k B(\mathbf{k}) = O(\epsilon), \tag{A 3}$$

$$\omega(\mathbf{k}) = \{(k/\mu) \tanh \mu k\}^{\frac{1}{2}}. \tag{A 4}$$

Let D denote the time-differentiation operator. Then $DA(\mathbf{k})$, $DB(\mathbf{k})$ are $O(\epsilon)$, since $A(\mathbf{k})$, $B(\mathbf{k})$ change in time only as a result of nonlinear interactions, and the system is such that only forward-progressing waves are significant. When the Fourier series (A 1), (A 2) are substituted into (2.1 *c*), (2.1 *d*) and $O(\epsilon)$ terms retained explicitly, a pair of equations for $DA(\mathbf{k})$, $DB(\mathbf{k})$, $A(\mathbf{k})$, $B(\mathbf{k})$ is obtained. The quadratic terms of $O(\epsilon)$ may be expressed solely in terms of amplitudes A by use of (A 3). The differential equation for $A(\mathbf{k})$, found by elimination of $DB(\mathbf{k})$, $B(\mathbf{k})$ from the linear terms of the above pair of equations, is

$$\begin{aligned} &D^2A(\mathbf{k}) - 2i\omega(\mathbf{k})\,DA(\mathbf{k}) \\ &\quad = -\tfrac{1}{2}\epsilon \sum_{\mathbf{l}} \{\omega(\mathbf{k}-\mathbf{l}) + \omega(\mathbf{l}) + \omega(\mathbf{k})\}\, R(\mathbf{k}, -\mathbf{l})\, A(\mathbf{l})\, A(\mathbf{k}-\mathbf{l}) \times \exp[-i\{\omega(\mathbf{k}-\mathbf{l}) \\ &\qquad + \omega(\mathbf{l}) - \omega(\mathbf{k})\}t] - \epsilon \sum_{\mathbf{l}} \{\omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l}) + \omega(\mathbf{k})\}\, R(\mathbf{k},\mathbf{l})\, A^*(\mathbf{l})\, A(\mathbf{k}+\mathbf{l}) \\ &\qquad \times \exp[-i\{\omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l}) - \omega(\mathbf{k})\}t] + O(\epsilon^2), \end{aligned} \qquad \text{(A 5)}$$

where

$$R(\mathbf{k},\mathbf{l}) = \frac{\{\omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l})\}\{\mathbf{k}.\mathbf{l}\,\omega(\mathbf{k}+\mathbf{l}) + \mathbf{k}.(\mathbf{k}+\mathbf{l})\,\omega(\mathbf{l})\} + \omega^2(\mathbf{k})\,\mathbf{l}.(\mathbf{k}+\mathbf{l})}{2\omega(\mathbf{l})\,\omega(\mathbf{k}+\mathbf{l})\,\{\omega(\mathbf{k}) + \omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l})\}} - \frac{\mu^2\omega^2(\mathbf{k})\,\{\omega^2(\mathbf{l}) - \omega(\mathbf{l})\,\omega(\mathbf{k}+\mathbf{l}) + \omega^2(\mathbf{k}+\mathbf{l})\}}{2\{\omega(\mathbf{k}) + \omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l})\}},$$

and $\omega(-\mathbf{l})$ is to be interpreted as $-\omega(\mathbf{l})$ in calculating $R(\mathbf{k}, -\mathbf{l})$. (There is an incorrect sign in equation (2.3) for $R(\mathbf{k},\mathbf{l})$ in Bryant (1978).)

It is noted that the solution for the linear terms in (A 5) is

$$A(\mathbf{k}) = c_1 + c_2 \exp\{2i\omega(\mathbf{k})\,t\},$$

where c_1, c_2 are arbitrary constants; which, when substituted into equation (A 1), expresses η as a sum of forward- and backward-progressing waves. The integrating factor for (A 5) is $\exp\{-2i\omega(\mathbf{k})\,t\}$, yielding

$$\begin{aligned} DA(\mathbf{k}) = &-\tfrac{1}{2}i\epsilon \sum_{\mathbf{l}} R(\mathbf{k}, -\mathbf{l})\, A(\mathbf{l})\, A(\mathbf{k}-\mathbf{l}) \exp[-i\{\omega(\mathbf{k}-\mathbf{l}) + \omega(\mathbf{l}) - \omega(\mathbf{k})\}t] \\ &- i\epsilon \sum_{\mathbf{l}} R(\mathbf{k},\mathbf{l})\, A^*(\mathbf{l})\, A(\mathbf{k}+\mathbf{l}) \exp[-i\{\omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l}) - \omega(\mathbf{k})\}t] + O(\epsilon^2) \end{aligned} \qquad \text{(A 6)}$$

for forward-progressing waves. It is inconsistent to integrate again since

$$\omega(\mathbf{k}-\mathbf{l}) + \omega(\mathbf{l}) - \omega(\mathbf{k}) \quad \text{and} \quad \omega(\mathbf{k}+\mathbf{l}) - \omega(\mathbf{l}) - \omega(\mathbf{k})$$

are of a magnitude comparable with ϵ for some $\mathbf{k}$ and $\mathbf{l}$.

Equations (2.2) and (2.3) are related to (A 1) and (A 6) respectively by

$$A(\mathbf{k}) = a(\mathbf{k}) \exp i\{\omega(\mathbf{k}) - k_1 c\}t, \qquad \text{(A 7)}$$

where all $a(\mathbf{k})$ are constants. The conditions of the derivation of (A 6) are satisfied by (A 7) since $\omega(\mathbf{k}) - k_1 c$ is $O(\epsilon)$ for the dominant harmonics with $a(\mathbf{k}) = O(1)$, these being the harmonics generated near resonance, while away from resonance $\omega(\mathbf{k}) - k_1 c$ is $O(1)$ with $a(\mathbf{k}) = O(\epsilon)$.

REFERENCES

BENNEY, D. J. & LUKE, J. C. 1964 On the interactions of permanent waves of finite amplitude. *J. Math. & Phys.* **43**, 309–313.

BRYANT, P. J. 1978 Oblique instability of periodic waves in shallow water. *J. Fluid Mech.* **86**, 783–792.

JOHNSON, R. S. 1980 Water waves and Korteweg–de Vries equations. *J. Fluid Mech.* **97**, 701–719.

KADOMTSEV, B. B. & PETVIASHVILI, V. I. 1970 On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl.* **15**, 539–541.

KRICHEVER, I. M. & NOVIKOV, S. P. 1978 Holomorphic bundles over Riemann surfaces and the Kadomtsev–Petviashvili equation. *Func. Anal. Appl.* **12**, 276–286.

MCLEAN, JOHN W. 1982 Instabilities of finite-amplitude gravity waves on water of finite depth. *J. Fluid Mech.* **114**, 331–341.

MELVILLE, W. K. 1980 On the Mach reflexion of a solitary wave. *J. Fluid Mech.* **98**, 285–297.

MILES, J. W. 1977*a* Obliquely interacting solitary waves. *J. Fluid Mech.* **79**, 157–169.

MILES, J. W. 1977*b* Resonantly interacting solitary waves. *J. Fluid Mech.* **79**, 171–179.

MILES, J. W. 1980 Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11–43.